

Probabilistic representations of solutions of elliptic boundary value problem and non-symmetric semigroups

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Abstract

In this paper, we use a probabilistic approach to show that there exists a unique, bounded continuous solution to the Dirichlet boundary value problem for a general class of second order non-symmetric elliptic operators L with singular coefficients, which does not necessarily have the maximum principle. The theory of Dirichlet forms and heat kernel estimates play a crucial role in our approach. A probabilistic representation of the non-symmetric semigroup $\{T_t\}_{t \geq 0}$ generated by L is also given.

Keywords: Dirichlet boundary value problem, singular coefficient, non-symmetric semigroup, probabilistic representation, Dirichlet form, heat kernel estimate.

1 Introduction and the Main Theorem

In this paper, we will use probabilistic methods to study the Dirichlet boundary value problem for second order elliptic differential operators:

$$\begin{cases} Lu = 0 & \text{in } D \\ u = f & \text{on } \partial D, \end{cases} \quad (1.1)$$

where D is a bounded connected open subset of \mathbb{R}^d . The operator L is given by

$$Lu = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + (c(x) - \operatorname{div} \hat{b}(x))u, \quad (1.2)$$

where $A(x) = (a_{ij}(x))_{i,j=1}^d$ is a Borel measurable, (not necessarily symmetric) matrix-valued function on D satisfying

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \quad \text{for any } \xi = (\xi_i)_{i=1}^d \in \mathbb{R}^d, x \in D \quad (1.3)$$

and

$$|a_{ij}(x)| \leq \frac{1}{\lambda} \quad \text{for any } x \in D, 1 \leq i, j \leq d \quad (1.4)$$

for some constant $0 < \lambda \leq 1$; $b = (b_1, \dots, b_d)^*$ and $\hat{b} = (\hat{b}_1, \dots, \hat{b}_d)^*$ are Borel measurable \mathbb{R}^d -valued functions on D and c is a Borel measurable function on D satisfying $|b|^2 \in L^{p \vee 1}(D; dx)$, $|\hat{b}|^2 \in L^{p \vee 1}(D; dx)$ and $c \in L^{p \vee 1}(D; dx)$ for some constant $p > d/2$. Hereafter we use $*$ to denote the transpose of a vector or matrix, and use $|\cdot|$ and $\langle \cdot, \cdot \rangle$ to denote respectively the standard norm and inner product of the Euclidean space \mathbb{R}^d .

In (1.1), $Lu = 0$ in D is understood in the distributional sense:

$$u \in H^{1,2}(D) \text{ and } \mathcal{E}(u, \phi) = 0 \text{ for every } \phi \in C_0^\infty(D),$$

where $H^{1,2}(D)$ is the Sobolev space on D with norm

$$\|f\|_{H^{1,2}} := \left(\int_D |\nabla f(x)|^2 dx + \int_D |f(x)|^2 dx \right)^{1/2},$$

$C_0^\infty(D)$ is the space of infinitely differentiable functions with compact support in D , and $(\mathcal{E}, D(\mathcal{E}))$ is the bilinear form associated with L :

$$\begin{aligned} \mathcal{E}(u, v) &= \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^d \int_D b_i(x) \frac{\partial u}{\partial x_i} v(x) dx \\ &\quad - \sum_{i=1}^d \int_D \hat{b}_i(x) \frac{\partial(uv)}{\partial x_i} dx - \int_D c(x) u(x) v(x) dx, \\ D(\mathcal{E}) &= H_0^{1,2}(D) \end{aligned} \quad (1.5)$$

with $H_0^{1,2}(D)$ being the completion of $C_0^\infty(D)$ with respect to the Sobolev-norm $\|\cdot\|_{H^{1,2}}$. By setting $a = I$, $b = 0$, $\hat{b} = 0$ and $c = 0$ off D , we may assume that the operator L is defined on \mathbb{R}^d .

Using probabilistic approaches to solve boundary value problems has a long history. The pioneering work goes back to Kakutani [10], who used Brownian motion to represent the solution of the classical Dirichlet boundary value problem with operator $L = \Delta$, the Laplacian operator. If $\hat{b} = 0$ and $c \leq 0$, then the solution u to problem (1.1) is given by the famous Feynman-Kac formula

$$u(x) = E_x \left[e^{\int_0^{\tau_D} c(X_s) ds} f(X_{\tau_D}) \right], \quad x \in D,$$

where $X = (X_t)_{t \geq 0}$ is the diffusion process associated with the generator L^b given by

$$L^b u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i}, \quad (1.6)$$

and τ_D is the first exit time of X from D . We refer the readers to [6] for the general results obtained in this case.

When $\hat{b} \neq 0$ and A is symmetric, Chen and Zhang [4] used the time reversal of symmetric Markov processes to give an explicit probabilistic representation of the solution to problem (1.1). (Note that the operator L given by (1.2) is the same as that used in [4] if we replace b with $b - \hat{b}$ in (1.2).) We should point out that the $\text{div } \hat{b}$ in (1.2) is just a formal writing since the vector field \hat{b} is merely measurable hence its divergence exists only in the distributional sense. In the remarkable paper [4], Chen and Zhang proved that there exists a unique, bounded continuous weak solution to problem (1.1) without the Markov assumption

$$c - \text{div } \hat{b} \leq 0 \quad \text{in } \mathbb{R}^d, \quad (1.7)$$

i.e., $\int_{\mathbb{R}^d} c(x) \phi(x) dx + \sum_{i=1}^d \int_{\mathbb{R}^d} \hat{b}_i(x) \frac{\partial \phi}{\partial x_i} dx \leq 0$ for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^d)$. The novelty of [4] is to tackle the lower-order term $\text{div } \hat{b}$ through combining the time-reversal of a Girsanov transform from the random time τ_D with a certain h -transform. In [4], Chen and Zhang used essentially the following result due to Meyers [17]:

For every $x_0 \in \mathbb{R}^d$, $R > 0$ and $p > d$, there is a constant $\varepsilon \in (0, 1)$, depending only on d , R and p , such that if

$$(1 - \varepsilon) I_{d \times d} \leq A(x) \leq I_{d \times d} \quad \text{for a.e. } x \in B_R := B(x_0, R), \quad (1.8)$$

then

$$\frac{1}{2} \nabla (A \nabla u) = \text{div } f \quad (1.9)$$

in B_R has a unique weak solution in $H_0^{1,p}(B_R)$ for every $f = (f_1, \dots, f_d) \in L^p(B_R; dx)$. Moreover, there is a constant $c > 0$ independent of f such that

$$\|\nabla u\|_{L^p(B_R; dx)} \leq c \|f\|_{L^p(B_R; dx)}.$$

To apply Meyers's result, the diffusion matrix A is assumed to satisfy Condition (1.8) in [4] (see [4, Theorems 3.3 and 4.5]). If Condition (1.8) is replaced with other conditions which guarantee that ∇u in (1.9) belongs to some L^p space for $p > d$, e.g. the condition that A is in the class VMO and $\partial D \in C^{1,1}$ (see [8]), then Chen and Zhang's approach still apply. We thank Professors Z.Q. Chen and T.S. Zhang for pointing out this to us.

In general, it is possible that $f \in L^p$ while $\nabla u \notin L^p$ (see [17] for an example). For this case, we cannot use the h -transform method to tackle the lower-order term $\operatorname{div} \hat{b}$ even when A is symmetric. In this paper, we will show that there exists a unique, bounded continuous solution to problem (1.1) without additional condition on A such as Condition (1.8), the VMO condition or the symmetry of A , and without the Markovian assumption (1.7). Instead of using Meyers's L^p -estimate as in [4], we will make use of Aronson's heat kernel estimates [1, 2].

In the sequel, we let $X = ((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$ be the Markov process associated with the following (non-symmetric) Dirichlet form

$$\begin{aligned} \mathcal{E}^0(u, v) &= \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx, \\ D(\mathcal{E}^0) &= H^{1,2}(\mathbb{R}^d). \end{aligned} \quad (1.10)$$

It is well-known that X is a conservative Feller process on \mathbb{R}^d that has continuous transition density function which admits a two-sided Aronson's heat kernel estimate. Let $\{\mathcal{F}_t, t \geq 0\}$ be the minimal augmented filtration generated by X . By Fukushima's decomposition (cf. [22, Theorem 5.1.8]), we have

$$X_t = x + M_t + N_t,$$

where $M_t = (M_t^1, \dots, M_t^d)^*$ is a martingale additive functional of X with quadratic co-variation

$$\langle M^i, M^j \rangle_t = \int_0^t \tilde{a}_{ij}(X_s) ds$$

and $N_t = (N_t^1, \dots, N_t^d)^*$ is a continuous additive functional of X locally of zero quadratic variation. Hereafter $\tilde{A} = (\tilde{a}_{ij})_{i,j=1}^d$ denotes the symmetrization of A , i.e., $\tilde{A} := 1/2(A + A^*)$.

For any vector field $\xi \in L^2(\mathbb{R}^d; dx)$, there exists a unique function $\xi^H \in H^{1,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \langle \xi, \nabla h \rangle dx = -\mathcal{E}_1^0(\xi^H, h), \quad \forall h \in C_0^\infty(\mathbb{R}^d)$$

(see Lemma 2.2 below). Hereafter $\mathcal{E}_\gamma^0(u, v) := \mathcal{E}^0(u, v) + \gamma \int uv dx$ for any $u, v \in D(\mathcal{E}^0)$ and any constant γ . We have Fukushima's decomposition:

$$\widetilde{\xi^H}(X_t) - \widetilde{\xi^H}(X_0) = M_t^{\xi^H} + N_t^{\xi^H},$$

where $\widetilde{\xi^H}$ is a quasi-continuous version of ξ^H . To simplify notation, in the sequel we take w to be its quasi-continuous version \tilde{w} whenever such a version exists. As in [9, 16], we use the term “quasi-everywhere” (abbreviated “q.e.”) to mean “except on an exceptional set”.

Now we can state the main theorem of this paper.

Theorem 1.1. *Let $d \geq 1$, D be a bounded Lipschitz domain in \mathbb{R}^d and $p > d/2$. Suppose that*

- (i) *A satisfies (1.3) and (1.4).*
- (ii) *$|b|^2 \in L^{p\vee 1}(D; dx)$ and $|\hat{b}|^2 \in L^{p\vee 1}(D; dx)$.*
- (iii) *$c \in L^{p\vee 1}(D; dx)$ and $c - \operatorname{div} \hat{b} \leq g$ for some nonnegative function $g \in L^{p\vee 1}(D; dx)$ in the distributional sense.*

Then, there exists a constant $M > 0$ such that whenever $\|g\|_{L^{p\vee 1}} \leq M$, for any $f \in C(\partial D)$, there exists a unique weak solution u to $Lu = 0$ in D that is continuous on \overline{D} with $u = f$ on ∂D . Moreover, the solution u admits the following representation: for q.e. $x \in D$,

$$\begin{aligned} u(x) = E_x \left[\exp \left(\int_0^{\tau_D} (\tilde{a}^{-1}b)^*(X_s) dM_s - \frac{1}{2} \int_0^{\tau_D} b^* \tilde{a}^{-1} b(X_s) ds \right. \right. \\ \left. \left. + \int_0^{\tau_D} c(X_s) ds + N_{\tau_D}^{\hat{b}^H} - \int_0^{\tau_D} \hat{b}^H(X_s) ds \right) f(X_{\tau_D}) \right]. \end{aligned} \quad (1.11)$$

We will give the proof of Theorem 1.1 in Section 2, which consists of three subsections. In Subsection 2.1, we prove the existence of the weak solution and give its probabilistic representation (1.11). In Subsection 2.2, we prove the continuity of the weak solution. In Subsection 2.3, we prove the uniqueness of the continuous weak solutions. The recently developed Nakao integral for non-symmetric Dirichlet forms (cf. [25] and [3]) will be used in the proof of the uniqueness.

In Section 3, we use some techniques of Section 2 to give a probabilistic representation of the non-symmetric semigroup $\{T_t\}_{t \geq 0}$ generated by L that is defined by (1.2). The obtained result (see Theorem 3.1 below) generalizes the corresponding result of [15] from the case of symmetric diffusion matrix A to the non-symmetric case.

2 Proof of Theorem 1.1

2.1 Proof of the existence of weak solution

We first generalize [6, Theorem 1.1] from the case of symmetric diffusion matrix A to the non-symmetric case. Define

$$L^1 u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + c(x)u.$$

Lemma 2.1. *Suppose that D is a bounded domain in \mathbb{R}^d , $c \leq 0$ and $f \in C(\partial D)$. Then*

$$\begin{aligned} u(x) = E_x \left[\exp \left(\int_0^{\tau_D} (\tilde{a}^{-1}b)^*(X_s) dM_s - \frac{1}{2} \int_0^{\tau_D} b^* \tilde{a}^{-1} b(X_s) ds \right. \right. \\ \left. \left. + \int_0^{\tau_D} c(X_s) ds \right) f(X_{\tau_D}) \right] \end{aligned}$$

is the unique weak solution of $L^1 u = 0$ which is continuous in D and

$$\lim_{x \rightarrow y, x \in D} u(x) = f(y)$$

for $y \in \partial D$ which is regular for the Laplace operator $(\frac{1}{2}\Delta, D)$.

Proof. The proof of Lemma 2.1 is similar to that of [6, Theorem 1.1]. We only point out below the main differences in the argument between the symmetric and the non-symmetric cases.

Denote by X^0 the part of the process X on D , that is, X^0 is obtained by killing the sample paths of X upon leaving D . By [1, 2], the transition density function $p_0(t, x, y)$ of X^0 has the upbound estimate

$$p_0(t, x, y) \leq \frac{\vartheta}{t^{d/2}} e^{-\frac{|x-y|^2}{\vartheta t}}, \quad (t, x, y) \in (0, \infty) \times D \times D, \quad (2.1)$$

for some constant $\vartheta > 0$.

We define

$$L^0 u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right).$$

Let D_1 be a bounded subdomain of D and $f_1 \in H_0^{1,2}(D)$. By [24], there exists a unique weak solution of $L^0 u = 0$ in D_1 such that $u - f_1|_{D_1} \in H_0^{1,2}(D_1)$. Further, by the famous theorem of Littman, Stampacchia and Weinberger, which holds also for the non-symmetric case (cf. e.g. [12]), we can prove the analog of [6, Theorem 2.1] with the non-symmetric A . By virtue of the Harnack inequality for parabolic

equations (cf. [20] and [14]), we can prove that [6, Lemma 2.2] and hence [6, Corollary 2.3 and Theorem 2.4] hold for the non-symmetric case.

Finally, we would like to point out that the exponential martingale M_t introduced in [6, (3.4)] needs to be replaced with

$$U_t := \exp \left(\int_0^t (\tilde{a}^{-1}b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds \right), \quad t \geq 0 \quad (2.2)$$

for our non-symmetric case. \square

Lemma 2.2. (i) For any vector field $\xi \in L^2(\mathbb{R}^d; dx)$, there exists a unique function $\xi^H \in H^{1,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \langle \xi, \nabla h \rangle dx = -\mathcal{E}_1^0(\xi^H, h), \quad \forall h \in H^{1,2}(\mathbb{R}^d). \quad (2.3)$$

(ii) If ξ_n converges to ξ in $L^2(\mathbb{R}^d; dx)$ as $n \rightarrow \infty$, then ξ_n^H converges to ξ^H in $H^{1,2}(\mathbb{R}^d)$ as $n \rightarrow \infty$.

(iii) For $\xi \in C_0^\infty(\mathbb{R}^d)$,

$$-\int_0^t \operatorname{div} \xi(X_s) ds = N_t^{\xi^H} - \int_0^t \xi^H(X_s) ds, \quad t \geq 0. \quad (2.4)$$

Proof. (i) Let $\xi \in L^2(\mathbb{R}^d; dx)$. We define the map $\eta : h \in H^{1,2}(\mathbb{R}^d) \mapsto \int_{\mathbb{R}^d} \langle \xi, \nabla h \rangle dx$. By the Riesz representation theorem, there exists a unique $\xi^0 \in H^{1,2}(\mathbb{R}^d)$ such that

$$\eta(h) = \tilde{\mathcal{E}}_1^0(\xi^0, h), \quad \forall h \in H^{1,2}(\mathbb{R}^d), \quad (2.5)$$

where $(\tilde{\mathcal{E}}^0, D(\mathcal{E}^0))$ denotes the symmetric part of the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$. Thus, by [3, Lemma 2.1], there exists a unique $\xi^H \in D(\mathcal{E}^0) = H^{1,2}(\mathbb{R}^d)$ such that

$$\tilde{\mathcal{E}}_1^0(\xi^0, h) = -\mathcal{E}_1^0(\xi^H, h), \quad \forall h \in H^{1,2}(\mathbb{R}^d). \quad (2.6)$$

(ii) Suppose ξ_n converges to ξ in $L^2(\mathbb{R}^d; dx)$ as $n \rightarrow \infty$. By (2.5), we get

$$\begin{aligned} \|\xi_n^0 - \xi^0\|_{\tilde{\mathcal{E}}_1^0} &= \sup_{\|h\|_{\tilde{\mathcal{E}}_1^0}=1} \tilde{\mathcal{E}}_1^0(\xi_n^0 - \xi^0, h) \\ &= \sup_{\|h\|_{\tilde{\mathcal{E}}_1^0}=1} \int_{\mathbb{R}^d} \langle \xi_n - \xi, \nabla h \rangle dx \\ &\leq \|\xi_n - \xi\|_{L^2} \sup_{\|h\|_{\tilde{\mathcal{E}}_1^0}=1} \|h\|_{H^{1,2}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (2.7)$$

Further, by (2.6), we get

$$\begin{aligned} \mathcal{E}_1^0(\xi_n^H - \xi^H, \xi_n^H) &= \mathcal{E}_1^0(\xi_n^H, \xi_n^H) - \mathcal{E}_1^0(\xi^H, \xi_n^H) \\ &= -\tilde{\mathcal{E}}_1^0(\xi_n^0, \xi_n^H) + \tilde{\mathcal{E}}_1^0(\xi^0, \xi_n^H) \\ &= \tilde{\mathcal{E}}_1^0(\xi^0 - \xi_n^0, \xi_n^H) \\ &\leq \left[\tilde{\mathcal{E}}_1^0(\xi^0 - \xi_n^0, \xi^0 - \xi_n^0) \right]^{1/2} \left[\tilde{\mathcal{E}}_1^0(\xi_n^H, \xi_n^H) \right]^{1/2}, \end{aligned} \quad (2.8)$$

$$\sup_{n \in \mathbb{N}} \mathcal{E}_1^0(\xi_n^H, \xi_n^H) \leq \sup_{n \in \mathbb{N}} \tilde{\mathcal{E}}_1^0(\xi_n^0, \xi_n^0) < \infty, \quad (2.9)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{E}_1^0(\xi_n^H - \xi^H, \xi^H) &= - \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_1^0(\xi_n^0 - \xi^0, \xi^H) \\ &= - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \langle \xi_n - \xi, \nabla \xi^H \rangle dx \\ &= 0. \end{aligned} \quad (2.10)$$

Therefore, we obtain by (2.7)-(2.10) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{E}_1^0(\xi_n^H - \xi^H, \xi_n^H - \xi^H) &= \lim_{n \rightarrow \infty} \{ \mathcal{E}_1^0(\xi_n^H - \xi^H, \xi_n^H) - \mathcal{E}_1^0(\xi_n^H - \xi^H, \xi^H) \} \\ &= 0. \end{aligned}$$

(iii) Let $\xi \in C_0^\infty(\mathbb{R}^d)$. For any $h \in H^{1,2}(\mathbb{R}^d)$, we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} E_{h \cdot dx} \left[- \int_0^t \operatorname{div} \xi(X_s) ds \right] &= - \int_{\mathbb{R}^d} (\operatorname{div} \xi) h dx \\ &= \int_{\mathbb{R}^d} \langle \xi, \nabla h \rangle dx \\ &= - \mathcal{E}_1^0(\xi^H, h) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} E_{h \cdot dx} \left[N_t^{\xi^H} - \int_0^t \xi^H(X_s) ds \right]. \end{aligned}$$

Therefore, (2.4) holds by [3, Lemma 2.3]. \square

Proof of the existence of weak solution and its probabilistic representation.

We define a family of measures $\{Q_x, x \in \mathbb{R}^d\}$ on \mathcal{F}_∞ by

$$\left. \frac{dQ_x}{dP_x} \right|_{\mathcal{F}_t} = U_t, \quad t \geq 0,$$

where U_t is given by (2.2). Then, under $\{Q_x, x \in \mathbb{R}^d\}$, X is a diffusion process on \mathbb{R}^d with the generator L^b given by (1.6). Denote by E_x^Q the expectation with respect to the measure Q_x for $x \in \mathbb{R}^d$. **From now on till the end of this section, we fix a constant $0 < \theta < \frac{1}{2}$.** We will show below that there exists a constant $M > 0$ such that for any $w \in \dot{L}^{p \vee 1}(\mathbb{R}^d; dx)$ with $\|w\|_{L^{p \vee 1}} \leq M$, we have

$$\sup_{x \in D} E_x^Q \left[\int_0^{\tau_D} |w|(X_s) ds \right] \leq \theta. \quad (2.11)$$

We only prove (2.11) when $d \geq 3$. The cases that $d = 1, 2$ can be considered similarly. Let X^D be the part of the process X on D under $\{Q_x\}$, that is, X^D is

obtained by killing the sample paths of X upon leaving D . Denote by $p(t, x, y)$ the transition density function of X^D . By [2, Theorem 9], for each $T > 0$, there exist positive constants σ_1^T and σ_2^T such that

$$p(t, x, y) \leq \frac{\sigma_1^T}{t^{d/2}} e^{-\frac{\sigma_2^T |x-y|^2}{t}}, \quad (t, x, y) \in (0, T) \times D \times D.$$

Similar to the proof of [13, Lemma 6.1], we can show that there exist positive constants σ_1 and σ_2 such that

$$p(t, x, y) \leq \frac{\sigma_1}{t^{d/2}} e^{-\frac{\sigma_2 |x-y|^2}{t}}, \quad (t, x, y) \in (0, \infty) \times D \times D. \quad (2.12)$$

Denote by $G_D(x, y)$ the Green function of X^D . Then,

$$G_D(x, y) \leq \frac{\sigma_3}{|x - y|^{d-2}}, \quad (x, y) \in D \times D, \quad (2.13)$$

for some positive constant σ_3 .

Let $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then $d - q(d - 2) > 0$. We obtain by (2.13) that

$$\begin{aligned} E_x^Q \left[\int_0^{\tau_D} |w|(X_s) ds \right] &= \int_D G_D(x, y) |w|(y) dy \\ &\leq \int_D \frac{\sigma_3 |w|(y)}{|x - y|^{d-2}} dy \\ &\leq \sigma_3 \left(\int_D (|w|(y))^p dy \right)^{1/p} \left(\int_D |x - y|^{-q(d-2)} dy \right)^{1/q} \\ &\leq \sigma_3 \|w\|_{L^p} \left(\int_0^\varsigma r^{d-q(d-2)-1} dr \right)^{1/q} \\ &= \frac{\sigma_3 \varsigma^{d/q-(d-2)}}{[d - q(d-2)]^{1/q}} \|w\|_{L^p}. \end{aligned}$$

Hereafter ς denotes the diameter of D . Set

$$M := \frac{\theta [d - q(d-2)]^{1/q}}{\sigma_3 \varsigma^{d/q-(d-2)}}.$$

Then $\|w\|_{L^p} \leq M$ implies (2.11). Further, by (2.11) and Khasminskii's inequality, we get

$$\sup_{x \in D} E_x^Q \left[\exp \left(\int_0^{\tau_D} |w|(X_s) ds \right) \right] \leq \frac{1}{1 - \theta}. \quad (2.14)$$

We define

$$J(x) = \frac{1_{\{|x| < 1\}} e^{-\frac{1}{1-|x|^2}}}{\int_{\{|y| < 1\}} e^{-\frac{1}{1-|y|^2}} dy}, \quad x \in \mathbb{R}^d.$$

For $k \in \mathbb{N}$ and $x \in \mathbb{R}^d$, set

$$\begin{aligned} J_k(x) &:= k^d J(kx), \\ \hat{b}_k(x) &:= \int_{\mathbb{R}^d} \hat{b}(x-y) J_k(y) dy, \\ c_k(x) &:= \int_{\mathbb{R}^d} c(x-y) J_k(y) dy, \\ g_k(x) &:= \int_{\mathbb{R}^d} g(x-y) J_k(y) dy. \end{aligned}$$

We have

$$\hat{b}_k \rightarrow \hat{b} \text{ in } L^2(\mathbb{R}^d; dx) \text{ as } k \rightarrow \infty \quad (2.15)$$

and

$$c_k \rightarrow c \text{ in } L^1(\mathbb{R}^d; dx) \text{ as } k \rightarrow \infty. \quad (2.16)$$

Suppose $\|g\|_{L^{p \vee 1}} \leq M$. Since $c - \operatorname{div} \hat{b} \leq g$ implies that $c_k - \operatorname{div} \hat{b}_k \leq g_k$ for $k \in \mathbb{N}$, we obtain by (2.14) that

$$\sup_{k \in \mathbb{N}} \sup_{x \in D} E_x^Q \left[\exp \left(\int_0^{\tau_D} (c_k - \operatorname{div} \hat{b}_k)(X_s) ds \right) \right] \leq \frac{1}{1-\theta}. \quad (2.17)$$

Define for $t \geq 0$,

$$\begin{aligned} Z_t &:= \exp \left(\int_0^t (\tilde{a}^{-1} b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds \right. \\ &\quad \left. + \int_0^t c(X_s) ds + N_t^{\hat{b}^H} - \int_0^t \hat{b}^H(X_s) ds \right). \end{aligned} \quad (2.18)$$

By (2.15) and Lemma 2.2(ii), we get

$$\hat{b}_k^H \rightarrow \hat{b}^H \text{ in } H^{1,2}(\mathbb{R}^d) \text{ as } k \rightarrow \infty. \quad (2.19)$$

Further, by [22, Lemma 4.1.12 and Theorem 5.1.2], there exists a subsequence $\{k_l\}$ such that for q.e. $x \in \mathbb{R}^d$,

$$P_x \left\{ \lim_{l \rightarrow \infty} N_t^{\hat{b}_{k_l}^H} = N_t^{\hat{b}^H} \text{ uniformly on any finite interval of } t \right\} = 1. \quad (2.20)$$

For simplicity, we still use $\{k\}$ to denote the subsequence $\{k_l\}$. By (2.16)-(2.20) and Fatou's lemma, we obtain that

$$\begin{aligned} E_x[Z_{\tau_D}] &= E_x^Q \left[\exp \left(\int_0^{\tau_D} c(X_s) ds + N_{\tau_D}^{\hat{b}^H} - \int_0^{\tau_D} \hat{b}^H(X_s) ds \right) \right] \\ &\leq \liminf_{k \rightarrow \infty} E_x^Q \left[\exp \left(\int_0^{\tau_D} c_k(X_s) ds + N_{\tau_D}^{\hat{b}_k^H} - \int_0^{\tau_D} \hat{b}_k^H(X_s) ds \right) \right] \\ &= \liminf_{k \rightarrow \infty} E_x^Q \left[\exp \left(\int_0^{\tau_D} (c_k - \operatorname{div} \hat{b}_k)(X_s) ds \right) \right] \\ &\leq \frac{1}{1-\theta}, \text{ for q.e. } x \in D. \end{aligned} \quad (2.21)$$

For $k \in \mathbb{N}$, we define

$$L_k u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + (c_k(x) - \operatorname{div} \hat{b}_k(x))u.$$

The bilinear form $(\mathcal{E}_k, D(\mathcal{E}_k))$ associated with L_k is

$$\begin{aligned} \mathcal{E}_k(u, v) &= \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^d \int_D b_i(x) \frac{\partial u}{\partial x_i} v(x) dx \\ &\quad - \sum_{i=1}^d \int_D \hat{b}_{k,i}(x) \frac{\partial(uv)}{\partial x_i} dx - \int_D c_k(x) u(x) v(x) dx, \\ D(\mathcal{E}_k) &= H_0^{1,2}(D). \end{aligned}$$

By (2.17), following the argument of [4, Theorem 4.3, pages 1030-1031], we can show that the weak solution to the Dirichlet boundary value problem

$$\begin{cases} L_k u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases} \quad (2.22)$$

is given by

$$\begin{aligned} u_k(x) &= E_x^Q \left[\exp \left(\int_0^{\tau_D} (c_k - \operatorname{div} \hat{b}_k)(X_s) ds \right) f(X_{\tau_D}) \right] \\ &= E_x \left[\exp \left(\int_0^{\tau_D} (\tilde{a}^{-1} b)^*(X_s) dM_s - \frac{1}{2} \int_0^{\tau_D} b^* \tilde{a}^{-1} b(X_s) ds \right. \right. \\ &\quad \left. \left. + \int_0^{\tau_D} (c_k - \operatorname{div} \hat{b}_k)(X_s) ds \right) f(X_{\tau_D}) \right]. \end{aligned}$$

Denote by v the right-hand side of (1.11). We claim that

$$\lim_{k \rightarrow \infty} u_k(x) = v(x), \quad \text{for q.e. } x \in D. \quad (2.23)$$

In fact, define

$$\begin{aligned} W_k &:= \exp \left(\int_0^{\tau_D} (c_k - \operatorname{div} \hat{b}_k)(X_s) ds \right) \\ &= \exp \left(\int_0^{\tau_D} c_k(X_s) ds + N_{\tau_D}^{\hat{b}_k^H} - \int_0^{\tau_D} \hat{b}_k^H(X_s) ds \right), \quad k \in \mathbb{N}, \\ W &:= \exp \left(\int_0^{\tau_D} c(X_s) ds + N_{\tau_D}^{\hat{b}^H} - \int_0^{\tau_D} \hat{b}^H(X_s) ds \right). \end{aligned}$$

By (2.16), (2.19) and (2.20), we get $W_k \rightarrow W$ in probability under Q_x as $k \rightarrow \infty$ for q.e. $x \in D$. By (2.11) and Khasminskii's inequality, we obtain that for $x \in D$,

$$\sup_{k \in \mathbb{N}} E_x^Q [W_k^2] = \sup_{k \in \mathbb{N}} E_x^Q \left[\exp \left(2 \int_0^{\tau_D} (c_k - \operatorname{div} \hat{b}_k)(X_s) ds \right) \right]$$

$$\begin{aligned}
&\leq \sup_{k \in \mathbb{N}} E_x^Q \left[\exp \left(2 \int_0^{\tau_D} g_k(X_s) ds \right) \right] \\
&\leq \frac{1}{1 - 2\theta}.
\end{aligned} \tag{2.24}$$

Hence $\{W_k\}$ is uniformly integrable under Q_x for $x \in D$. Therefore, (2.23) holds.

Finally, we show that v is a weak solution to problem (1.1). By (2.24), we get

$$\begin{aligned}
\sup_{k \in \mathbb{N}} \|u_k\|_{L^2}^2 &= \sup_{k \in \mathbb{N}} \int_D (E_x^Q [W_k f(X_{\tau_D})])^2 dx \\
&< \frac{\|f\|_\infty^2 |D|}{1 - 2\theta},
\end{aligned} \tag{2.25}$$

where $|D|$ is the Lebesgue measure of D . Since u_k is the weak solution to problem (2.22), we have $\mathcal{E}_k(u_k, \phi) = 0$ for any $\phi \in C_0^\infty(D)$. Then, $\mathcal{E}_k(u_k, \phi) = 0$, $\forall \phi \in H_0^{1,2}(D)$. Thus, we have $\mathcal{E}_k(u_k, u_k - u_1) = 0$, which implies that

$$\mathcal{E}_k(u_k, u_k) = \mathcal{E}_k(u_k, u_1). \tag{2.26}$$

Note that $|b|^2$, $|\hat{b}|^2$ and c are in the Kato class. For any $0 < \varepsilon < 1$, there exists a constant $A(\varepsilon) > 1$ such that for $1 \leq i \leq d$ and $\eta \in H^{1,2}(\mathbb{R}^d)$ (cf. [11]),

$$\int_{\mathbb{R}^d} (b_i^2 + \hat{b}_i^2 + |c|) \eta^2 dx \leq \varepsilon \int_{\mathbb{R}^d} |\nabla \eta|^2 dx + A(\varepsilon) \int_{\mathbb{R}^d} \eta^2 dx. \tag{2.27}$$

By (2.27), we obtain that for $k \in \mathbb{N}$, $1 \leq i \leq d$ and $\eta \in H^{1,2}(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} ((\hat{b}_{k,i})^2 + |c_k|) \eta^2 dx \\
&\leq \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} [\hat{b}_i^2(x-y) + |c|(x-y)] J_k(y) dy \right\} \eta^2(x) dx \\
&\leq \varepsilon \int_{\mathbb{R}^d} |\nabla \eta|^2 dx + A(\varepsilon) \int_{\mathbb{R}^d} \eta^2 dx.
\end{aligned} \tag{2.28}$$

Then, we obtain by (2.26)-(2.28) that for $k \in \mathbb{N}$,

$$\begin{aligned}
\frac{\lambda}{2} \|\nabla u_k\|_{L^2}^2 &\leq \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} dx \\
&= \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u_k}{\partial x_i} \frac{\partial u_1}{\partial x_j} dx - \sum_{i=1}^d \int_D b_i(x) \frac{\partial u_k}{\partial x_i} u_1(x) dx \\
&\quad - \sum_{i=1}^d \int_D \hat{b}_{k,i}(x) \frac{\partial u_k}{\partial x_i} u_1(x) dx - \sum_{i=1}^d \int_D \hat{b}_{k,i}(x) u_k(x) \frac{\partial u_1}{\partial x_i} dx \\
&\quad - \int_D c_k(x) u_k(x) u_1(x) dx + \sum_{i=1}^d \int_D b_i(x) \frac{\partial u_k}{\partial x_i} u_k(x) dx
\end{aligned}$$

$$\begin{aligned}
& +2 \sum_{i=1}^d \int_D \hat{b}_{k,i}(x) \frac{\partial u_k}{\partial x_i} u_k(x) dx + \int_D c_k(x) u_k^2(x) dx \\
& \leq \frac{d^2}{2\lambda} \|\nabla u_k\|_{L^2} \|\nabla u_1\|_{L^2} + 2dA^{1/2}(\varepsilon) \|\nabla u_k\|_{L^2} \|u_1\|_{H^{1,2}} \\
& \quad + dA^{1/2}(\varepsilon) \|\nabla u_1\|_{L^2} \|u_k\|_{H^{1,2}} + A(\varepsilon) \|u_k\|_{H^{1,2}} \|u_1\|_{H^{1,2}} \\
& \quad + 3d \|\nabla u_k\|_{L^2} (\varepsilon \|\nabla u_k\|_{L^2}^2 + A(\varepsilon) \|u_k\|_{L^2}^2)^{1/2} \\
& \quad + (\varepsilon \|\nabla u_k\|_{L^2}^2 + A(\varepsilon) \|u_k\|_{L^2}^2). \tag{2.29}
\end{aligned}$$

Let ε be much smaller than λ . Then, we obtain by (2.25) and (2.29) that $\sup_{k \in \mathbb{N}} \|\nabla u_k\|_{L^2} < \infty$ and thus

$$\sup_{k \in \mathbb{N}} \|u_k\|_{H^{1,2}} < \infty.$$

By taking a subsequence if necessary, we may assume that $u_k \rightarrow v_1$ weakly in $H^{1,2}(D)$ as $k \rightarrow \infty$ and that its Cesaro mean $\{u'_k := \frac{1}{k} \sum_{l=1}^k u_l, k \geq 1\} \rightarrow v_2$ in $H^{1,2}(D)$ as $k \rightarrow \infty$. By (2.23) and [16, Proposition III.3.5], we obtain that $v_1(x) = v_2(x) = v(x)$ for q.e. $x \in D$ and

$$v \text{ is quasi-continuous in } D. \tag{2.30}$$

Let $\phi \in C_0^\infty(D)$. Note that for $l \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{E}_l(u_l, \phi) &= \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial u_l}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx - \sum_{i=1}^d \int_{\mathbb{R}^d} b_i(x) \frac{\partial u_l}{\partial x_i} \phi(x) dx \\
&\quad - \sum_{i=1}^d \int_{\mathbb{R}^d} \hat{b}_{l,i}(x) \frac{\partial (u_l \phi)}{\partial x_i} dx - \int_{\mathbb{R}^d} c_l(x) u_l(x) \phi(x) dx. \tag{2.31}
\end{aligned}$$

By (2.27) and (2.28), we find that (cf. [4, Lemma 2.2(iv)])

$$\lim_{k \rightarrow \infty} \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial u'_k}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx = \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx, \tag{2.32}$$

$$\lim_{k \rightarrow \infty} \sum_{i=1}^d \int_{\mathbb{R}^d} b_i(x) \frac{\partial u'_k}{\partial x_i} \phi(x) dx = \sum_{i=1}^d \int_{\mathbb{R}^d} b_i(x) \frac{\partial v}{\partial x_i} \phi(x) dx, \tag{2.33}$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{l=1}^k \sum_{i=1}^d \int_{\mathbb{R}^d} \hat{b}_{l,i}(x) \frac{\partial (u_l \phi)}{\partial x_i} dx = \sum_{i=1}^d \int_{\mathbb{R}^d} \hat{b}_i(x) \frac{\partial (v \phi)}{\partial x_i} dx, \tag{2.34}$$

and

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{l=1}^k \int_{\mathbb{R}^d} c_l(x) u_l(x) \phi(x) dx = \int_{\mathbb{R}^d} c(x) v(x) \phi(x) dx. \tag{2.35}$$

Therefore, we obtain by (1.5) and (2.31)-(2.35) that $\mathcal{E}(v, \phi) = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{l=1}^k \mathcal{E}_l(u_l, \phi) = 0$. \square

2.2 Proof of the continuity of weak solution

It is well-known that any weak solution to $Lu = 0$ in D has a locally Hölder continuous version (see [18], cf. also [19]). Denote by v the right-hand side of (1.11) and denote by v^* its continuous version in D . We will show below that

$$\lim_{x \rightarrow y, x \in D} v^*(x) = f(y), \quad \forall y \in \partial D. \quad (2.36)$$

First, we prove an important lemma based on the Dirichlet heat kernel estimates obtained by Aronson.

Suppose $d \geq 2$. Let $p_1 > d$ and $q_1 > 1$ satisfy $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then $q_1 = \frac{p_1}{p_1-1} < \frac{d}{d-1}$. We choose $0 < \alpha < 1$ such that

$$q_1 < \frac{d}{d-\alpha}. \quad (2.37)$$

Let M_1 be a constant satisfying

$$e^{|x|} \geq M_1 |x|^{(d-\alpha+1)/2}, \quad \forall x \in \mathbb{R}^d. \quad (2.38)$$

Let $p_2 > d/2$ and $q_2 > 1$ satisfy $\frac{1}{p_2} + \frac{1}{q_2} = 1$. Then $q_2 = \frac{p_2}{p_2-1} < \frac{d}{d-2}$. We choose β satisfying

$$\frac{d}{2} - 1 < \beta < \frac{d}{2q_2}. \quad (2.39)$$

Let M_2 be a constant satisfying

$$e^{|x|} \geq M_2 |x|^\beta, \quad \forall x \in \mathbb{R}^d, \quad (2.40)$$

and let M_3 be a constant satisfying

$$e^{|x|} \geq M_3 |x|^{5/8}, \quad \forall x \in \mathbb{R}^d.$$

We denote by ς the diameter of D as above. By (2.37) and (2.39), we find that

$$\int_0^\varsigma r^{d-q_1(d-\alpha)-1} dr < \infty \quad \text{and} \quad \int_0^\varsigma r^{d-2\beta q_2-1} dr < \infty.$$

Denote

$$h(t, x, y) = \frac{\sigma_1}{t^{d/2}} e^{-\frac{\sigma_2 |x-y|^2}{t}}, \quad (t, x, y) \in (0, \infty) \times D \times D.$$

Then, we obtain by (2.12) that

$$p(t, x, y) \leq h(t, x, y), \quad (t, x, y) \in (0, \infty) \times D \times D.$$

Lemma 2.3. Let μ be a vector field on \mathbb{R}^d and ν be a function on \mathbb{R}^d such that $\mu, \nu \in C^\infty(\mathbb{R}^d)$.

(i) Suppose $d \geq 2$, $p_1 > d$ and $p_2 > d/2$. Then, for $t > 0$ and $x \in D$,

$$\begin{aligned} & \left| \int_{y \in D} h(t, x, y) \operatorname{div} \mu(y) dy \right| \\ & \leq \frac{2\sigma_1}{\sigma_2^{(d-\alpha-1)/2} M_1 t^{(1+\alpha)/2}} \left(\int_0^\varsigma r^{d-q_1(d-\alpha)-1} dr \right)^{1/q_1} \left(\int_{y \in D} |\mu(y)|^{p_1} dy \right)^{1/p_1} \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{y \in D} h(t, x, y) \nu(y) dy \right| \\ & \leq \frac{\sigma_1}{\sigma_2^\beta M_2 t^{d/2-\beta}} \left(\int_0^\varsigma r^{d-2\beta q_2-1} dr \right)^{1/q_2} \left(\int_{y \in D} |\nu(y)|^{p_2} dy \right)^{1/p_2}. \end{aligned}$$

(ii) Suppose $d = 1$. Then, for $t > 0$ and $x \in D$,

$$\left| \int_{y \in D} h(t, x, y) \operatorname{div} \mu(y) dy \right| \leq \frac{2^{3/2} \sigma_1 \sigma_2^{3/8} \varsigma^{1/4}}{M_3 t^{7/8}} \left(\int_{y \in D} |\mu(y)|^2 dy \right)^{1/2}$$

and

$$\left| \int_{y \in D} h(t, x, y) \nu(y) dy \right| \leq \frac{\sigma_1}{t^{1/2}} \int_{y \in D} |\nu(y)| dy.$$

Proof. We only prove (i). The proof of (ii) is similar so we omit it here.

By (2.38), we get

$$\begin{aligned} & \left| \int_{y \in D} h(t, x, y) \operatorname{div} \mu(y) dy \right| \\ & = \left| \int_{y \in D} \langle \nabla_y h(t, x, y), \mu(y) \rangle dy \right| \\ & \leq \int_{y \in D} \frac{\sigma_1}{t^{d/2} e^{\sigma_2 |x-y|^2/t}} \frac{2\sigma_2 |x-y|}{t} |\mu(y)| dy \\ & \leq \frac{2\sigma_1 \sigma_2}{t^{d/2+1}} \int_{y \in D} \frac{|x-y|}{M_1 (\sigma_2 |x-y|^2/t)^{(d-\alpha+1)/2}} |\mu(y)| dy \\ & \leq \frac{2\sigma_1}{\sigma_2^{(d-\alpha-1)/2} M_1 t^{(1+\alpha)/2}} \int_{y \in D} \frac{|\mu(y)|}{|x-y|^{d-\alpha}} dy \\ & \leq \frac{2\sigma_1}{\sigma_2^{(d-\alpha-1)/2} M_1 t^{(1+\alpha)/2}} \left(\int_{y \in D} \frac{1}{|x-y|^{q_1(d-\alpha)}} dy \right)^{1/q_1} \left(\int_{y \in D} |\mu(y)|^{p_1} dy \right)^{1/p_1} \\ & \leq \frac{2\sigma_1}{\sigma_2^{(d-\alpha-1)/2} M_1 t^{(1+\alpha)/2}} \left(\int_0^\varsigma r^{d-q_1(d-\alpha)-1} dr \right)^{1/q_1} \left(\int_{y \in D} |\mu(y)|^{p_1} dy \right)^{1/p_1}. \end{aligned}$$

By (2.40), we get

$$\begin{aligned}
& \left| \int_{y \in D} h(t, x, y) \nu(y) dy \right| \\
& \leq \int_{y \in D} \frac{\sigma_1}{t^{d/2} e^{\sigma_2 |x-y|^2/t}} |\nu(y)| dy \\
& \leq \int_{y \in D} \frac{\sigma_1}{M_2 t^{d/2} (\sigma_2 |x-y|^2/t)^\beta} |\nu(y)| dy \\
& = \frac{\sigma_1}{\sigma_2^\beta M_2 t^{d/2-\beta}} \int_{y \in D} \frac{|\nu(y)|}{|x-y|^{2\beta}} dy \\
& \leq \frac{\sigma_1}{\sigma_2^\beta M_2 t^{d/2-\beta}} \left(\int_{y \in D} \frac{1}{|x-y|^{2\beta q_2}} dy \right)^{1/q_2} \left(\int_{y \in D} |\nu(y)|^{p_2} dy \right)^{1/p_2} \\
& \leq \frac{\sigma_1}{\sigma_2^\beta M_2 t^{d/2-\beta}} \left(\int_0^\varsigma r^{d-2\beta q_2-1} dr \right)^{1/q_2} \left(\int_{y \in D} |\nu(y)|^{p_2} dy \right)^{1/p_2}.
\end{aligned}$$

□

Remark 2.4. In [7], Cho, Kim and Park established very nice sharp two-sided estimates on Dirichlet heat kernels. Under the additional assumption that D is a $C^{1,\alpha}$ -domain ($0 < \alpha \leq 1$) satisfying the connected line condition and each a_{ij} , $1 \leq i, j \leq d$, is Dini continuous, by [7, Theorem 1.1], for each $T > 0$, there exist positive constants c_1 and c_2 such that for $(t, x, y) \in (0, T) \times D \times D$,

$$p(t, x, y) \leq \left(1 \wedge \frac{\rho(x)}{\sqrt{t}} \right) \left(1 \wedge \frac{\rho(y)}{\sqrt{t}} \right) \frac{c_1}{t^{d/2}} e^{-\frac{c_2 |x-y|^2}{t}} \quad (2.41)$$

and

$$|\nabla_y p(t, x, y)| \leq \left(1 \wedge \frac{\rho(x)}{\sqrt{t}} \right) \frac{c_1}{t^{(d+1)/2}} e^{-\frac{c_2 |x-y|^2}{t}}, \quad (2.42)$$

where $\rho(x) := \text{dist}(x, \partial D)$.

By virtue of (2.41) and (2.42), we can obtain estimates for $p(t, x, y)$ similar to those for $h(t, x, y)$ given as in Lemma 2.3. These estimates for $p(t, x, y)$ or $h(t, x, y)$ make it possible to handle the case when Meyers's L^p -estimate is not available.

Proof of the continuity of weak solution at the boundary.

By (2.30), we have $v^*(x) = v(x)$ for q.e. $x \in D$. Note that for $x \in D$,

$$\begin{aligned}
v(x) &= E_x^Q \left[\exp \left(\int_0^{\tau_D} c(X_s) ds + N_{\tau_D}^{\hat{b}^H} - \int_0^{\tau_D} \hat{b}^H(X_s) ds \right) f(X_{\tau_D}) \right] \\
&= E_x^Q [f(X_{\tau_D})] + E_x^Q [f(X_{\tau_D})(e^{A_{\tau_D}} - 1)],
\end{aligned}$$

where $A_t := \int_0^t c(X_s) ds + N_t^{\hat{b}^H} - \int_0^t \hat{b}^H(X_s) ds$, $t \geq 0$. By Lemma 2.1, to prove (2.36), it suffices to show that there exists an exceptional set $F \subset D$ such that

$$\lim_{x \rightarrow y, x \in D \setminus F} E_x^Q [f(X_{\tau_D})(e^{A_{\tau_D}} - 1)] = 0, \quad \forall y \in \partial D. \quad (2.43)$$

For $t > 0$ and $x \in D$, we have

$$\begin{aligned} E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1)] &= E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1); \tau_D \leq t] \\ &\quad + E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1); \tau_D > t]. \end{aligned}$$

By (2.21), there exists an exceptional set $F_1 \subset D$ such that

$$\sup_{x \in D \setminus F_1} E_x^Q[\exp(A_{\tau_D})] = \sup_{x \in D \setminus F_1} E_x[Z_{\tau_D}] \leq \frac{1}{1 - \theta}.$$

Then, we obtain by the strong Markov property that for q.e. $x \in D$,

$$\begin{aligned} &|E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1); \tau_D > t]| \\ &\leq \|f\|_\infty \{Q_x(\tau_D > t) + E_x^Q[e^{A_{\tau_D}}; \tau_D > t]\} \\ &\leq \|f\|_\infty \left\{ Q_x(\tau_D > t) + \frac{E_x^Q[e^{A_{\tau_D}}; \tau_D > t]}{1 - \theta} \right\}, \quad \forall t > 0. \end{aligned} \quad (2.44)$$

By Lemma 2.1, following the argument of [6, (2.28)], we get

$$\lim_{x \rightarrow y, x \in D} Q_x(\tau_D > t) = 0, \quad \forall t > 0, \forall y \in \partial D. \quad (2.45)$$

By (2.16), (2.19), (2.20), (2.24) and Fatou's lemma, there exists an exceptional set $F_2 \subset D$ such that for every $t > 0$,

$$\sup_{x \in D \setminus F_2} E_x^Q[e^{2A_{\tau_D}}; \tau_D > t] \leq \sup_{x \in D \setminus F_2} \sup_{k \in \mathbb{N}} E_x^Q \left[e^{2 \int_0^{\tau_D} g_k(X_s) ds} \right] \leq \frac{1}{1 - 2\theta}. \quad (2.46)$$

Thus, we obtain by (2.44)-(2.46) that there exists an exceptional set $F_3 \subset D$ satisfying

$$\lim_{x \rightarrow y, x \in D \setminus F_3} E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1); \tau_D > t] = 0, \quad \forall t > 0, \forall y \in \partial D.$$

Therefore, to prove (2.43), it suffices to show that there exists an exceptional set $F_4 \subset D$ such that

$$\lim_{t \downarrow 0} \sup_{x \in D \setminus F_4} E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1); \tau_D \leq t] = 0. \quad (2.47)$$

By (2.16), (2.19), (2.20) and Fatou's lemma, there exists an exceptional set $F_4 \subset D$ such that for every $t > 0$,

$$\begin{aligned} &\sup_{x \in D \setminus F_4} |E_x^Q[f(X_{\tau_D})(e^{A_{\tau_D}} - 1); \tau_D \leq t]| \\ &\leq \|f\|_\infty \sup_{x \in D \setminus F_4} \liminf_{k \rightarrow \infty} E_x^Q \left[\left| e^{\int_0^{\tau_D} (c_k - \operatorname{div} \hat{b}_k)(X_s) ds} - 1 \right|; \tau_D \leq t \right] \\ &\leq \|f\|_\infty \left\{ \sup_{x \in D} \limsup_{k \rightarrow \infty} E_x^Q \left[e^{\int_0^{\tau_D} g_k(X_s) ds} - 1; \tau_D \leq t \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \sup_{x \in D} \limsup_{k \rightarrow \infty} E_x^Q \left[\left(1 - e^{\int_0^{\tau_D} (c_k - \operatorname{div} \hat{b}_k - g_k)(X_s) ds} \right); \tau_D \leq t \right] \Big\} \\
\leq & \|f\|_\infty \left\{ \sup_{x \in D} \limsup_{k \rightarrow \infty} E_x^Q \left[e^{\int_0^{t \wedge \tau_D} g_k(X_s) ds} - 1 \right] \right. \\
& \left. + \sup_{x \in D} \limsup_{k \rightarrow \infty} E_x^Q \left[\left(1 - e^{\int_0^{t \wedge \tau_D} (c_k - \operatorname{div} \hat{b}_k - g_k)(X_s) ds} \right) \right] \right\}.
\end{aligned}$$

By Lemma 2.3 and Khasminskii's inequality, we get

$$\lim_{t \downarrow 0} \sup_{x \in D} \sup_{k \in \mathbb{N}} E_x^Q \left[e^{\int_0^{t \wedge \tau_D} g_k(X_s) ds} \right] = 1.$$

Hence, to prove (2.47), we need only show that

$$\lim_{t \downarrow 0} \inf_{x \in D} \inf_{k \in \mathbb{N}} E_x^Q \left[e^{\int_0^{t \wedge \tau_D} (c_k - \operatorname{div} \hat{b}_k - g_k)(X_s) ds} \right] \geq 1.$$

Further, by Jensen's inequality, we need only show that

$$\lim_{t \downarrow 0} \sup_{x \in D} \sup_{k \in \mathbb{N}} E_x^Q \left[\int_0^{t \wedge \tau_D} (g_k - c_k + \operatorname{div} \hat{b}_k)(X_s) ds \right] = 0.$$

By Lemma 2.3, we obtain that

$$\begin{aligned}
& \sup_{x \in D} \sup_{k \in \mathbb{N}} E_x^Q \left[\int_0^{t \wedge \tau_D} (g_k - c_k + \operatorname{div} \hat{b}_k)(X_s) ds \right] \\
= & \sup_{x \in D} \sup_{k \in \mathbb{N}} \int_0^t \int_{y \in D} p(s, x, y) (g_k - c_k + \operatorname{div} \hat{b}_k)(y) dy ds \\
\leq & \sup_{x \in D} \sup_{k \in \mathbb{N}} \int_0^t \int_{y \in D} h(s, x, y) (g_k - c_k + \operatorname{div} \hat{b}_k)(y) dy ds \\
\rightarrow & 0 \text{ as } t \downarrow 0.
\end{aligned}$$

The proof is complete. \square

2.3 Proof of the uniqueness of continuous weak solutions

In this subsection, we will prove that there exists a unique continuous weak solution to problem (1.1).

Let u_1 be a weak solution to problem (1.1) such that u_1 is continuous on \overline{D} . We have Fukushima's decomposition

$$\begin{aligned}
u_1(X_t) - u_1(X_0) &= M_t^{u_1} + N_t^{u_1} \\
&= \int_0^t \nabla u_1(X_s) dM_s + N_t^{u_1}, \quad t < \tau_D.
\end{aligned} \tag{2.48}$$

We claim that

$$\begin{aligned}
N_t^{u_1} &= - \sum_{i=1}^d \int_0^t b_i(X_s) \frac{\partial u_1}{\partial x_i}(X_s) ds - \int_0^t u_1(X_s) c(X_s) ds \\
&\quad - \int_0^t u_1(X_s) dN_s^{\hat{b}^H} + \int_0^t u_1(X_s) \hat{b}^H(X_s) ds, \\
&\quad t < \tau_D, \quad P_x - \text{a.s. for q.e. } x \in D,
\end{aligned} \tag{2.49}$$

where the third term of (2.49) is a Nakao integral (we refer the readers to [3, Definition 2.4] and [21, Definition 3.1] for the definition).

Let $\{D_n\}$ be a sequence of increasing open subsets of \mathbb{R}^d satisfying $D = \cup_{n \in \mathbb{N}} D_n$ and $\overline{D_n} \subset D_{n+1}$ for each n . We choose a sequence $\{u^{(n)} \subset H_0^{1,2}(D) \cap \mathcal{B}_b(D_n)\}$ satisfying $u_1 = u^{(n)}$ on D_n for each n . To prove (2.49) it suffices to show that for any $n \in \mathbb{N}$,

$$\begin{aligned}
N_t^{u^{(n)}} &= - \sum_{i=1}^d \int_0^t b_i(X_s) \frac{\partial u^{(n)}}{\partial x_i}(X_s) ds - \int_0^t u^{(n)}(X_s) c(X_s) ds \\
&\quad - \int_0^t u^{(n)}(X_s) dN_s^{\hat{b}^H} + \int_0^t u^{(n)}(X_s) \hat{b}^H(X_s) ds, \\
&\quad t < \tau_{D_n}, \quad P_x - \text{a.s. for q.e. } x \in D.
\end{aligned} \tag{2.50}$$

Denote by $C_t^{(n)}$ the right hand side of (2.50). By [22, Theorem 5.2.7], following the argument of the proof of [21, Theorem 2.2], we find that to prove (2.50) it suffices to show that for each n ,

$$\lim_{t \downarrow 0} \frac{1}{t} E_{\phi \cdot dx} [N_t^{u^{(n)}}] = \lim_{t \downarrow 0} \frac{1}{t} E_{\phi \cdot dx} [C_t^{(n)}], \quad \forall \phi \in H_0^{1,2}(D_n) \cap \mathcal{B}_b(D_n). \tag{2.51}$$

We fix an $n \in \mathbb{N}$ and $\phi \in H_0^{1,2}(D_n) \cap \mathcal{B}_b(D_n)$. By (1.5), (1.10) and (2.3), we get

$$\begin{aligned}
\mathcal{E}^0(u^{(n)}, \phi) &= \mathcal{E}(u^{(n)}, \phi) + \sum_{i=1}^d \int_D b_i(x) \frac{\partial u^{(n)}}{\partial x_i} \phi(x) dx \\
&\quad + \sum_{i=1}^d \int_D \hat{b}_i(x) \frac{\partial (u^{(n)} \phi)}{\partial x_i} + \int_D c(x) u^{(n)}(x) \phi(x) dx, \\
&= \sum_{i=1}^d \int_D b_i(x) \frac{\partial u^{(n)}}{\partial x_i} \phi(x) dx + \int_D c(x) u^{(n)}(x) \phi(x) dx \\
&\quad - \mathcal{E}_1^0(\hat{b}^H, u^{(n)} \phi).
\end{aligned} \tag{2.52}$$

We have

$$\begin{aligned}
\lim_{t \downarrow 0} \frac{1}{t} E_{\phi \cdot dx} [N_t^{u^{(n)}}] &= \lim_{t \downarrow 0} \frac{1}{t} E_{\phi \cdot dx} [u^{(n)}(X_t) - u^{(n)}(X_0) - M_t^{u^{(n)}}] \\
&= \lim_{t \downarrow 0} \frac{1}{t} \int_D E_x [u^{(n)}(X_t) - u^{(n)}(X_0)] \phi(x) dx \\
&= -\mathcal{E}^0(u^{(n)}, \phi)
\end{aligned} \tag{2.53}$$

and

$$\begin{aligned}
& \lim_{t \downarrow 0} \frac{1}{t} E_{\phi \cdot dx} \left[- \sum_{i=1}^d \int_0^t b_i(X_s) \frac{\partial u^{(n)}}{\partial x_i}(X_s) ds - \int_0^t u^{(n)}(X_s) c(X_s) ds \right. \\
& \quad \left. + \int_0^t u^{(n)}(X_s) \hat{b}^H(X_s) ds \right] \\
&= - \sum_{i=1}^d \int_D b_i(x) \frac{\partial u^{(n)}}{\partial x_i} \phi(x) dx - \int_D c(x) u^{(n)}(x) \phi(x) dx \\
& \quad + \int_D \hat{b}^H(x) u^{(n)}(x) \phi(x) dx.
\end{aligned} \tag{2.54}$$

By [3, Remark 2.5], we get

$$\lim_{t \downarrow 0} \frac{1}{t} E_{\phi \cdot dx} \left[- \int_0^t u^{(n)}(X_s) dN_s^{\hat{b}^H} \right] = \mathcal{E}^0(\hat{b}^H, u^{(n)} \phi). \tag{2.55}$$

Then, (2.51) holds by (2.52)-(2.55). Thus, (2.50) and hence (2.49) hold.

By (2.48) and (2.49), we obtain that

$$\begin{aligned}
& u_1(X_t) - u_1(X_0) \\
&= \int_0^t \nabla u_1(X_s) dM_s - \sum_{i=1}^d \int_0^t b_i(X_s) \frac{\partial u_1}{\partial x_i}(X_s) ds \\
& \quad - \int_0^t u_1(X_s) c(X_s) ds - \int_0^t u_1(X_s) dN_s^{\hat{b}^H} + \int_0^t u_1(X_s) \hat{b}^H(X_s) ds, \\
& \quad t < \tau_D, \quad P_x - \text{a.s. for q.e. } x \in D.
\end{aligned} \tag{2.56}$$

We now prove that for $t < \tau_D$,

$$d(u_1(X_t) Z_t) = u_1(X_t) Z_t (\tilde{a}^{-1} b)^*(X_t) dM_t + Z_t \nabla u_1(X_t) dM_t, \tag{2.57}$$

$P_x - \text{a.s. for q.e. } x \in D$, where Z_t is defined as in (2.18).

For $k \in \mathbb{N}$ and $t > 0$, we define

$$\begin{aligned}
V_t^k &:= \int_0^t \nabla u_1(X_s) dM_s - \sum_{i=1}^d \int_0^t b_i(X_s) \frac{\partial u_1}{\partial x_i}(X_s) ds \\
& \quad - \int_0^t u_1(c_k - \operatorname{div} \hat{b}_k)(X_s) ds
\end{aligned} \tag{2.58}$$

and

$$\begin{aligned}
Z_t^k &:= \exp \left(\int_0^t (\tilde{a}^{-1} b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds \right. \\
& \quad \left. + \int_0^t (c_k - \operatorname{div} \hat{b}_k)(X_s) ds \right).
\end{aligned}$$

Then,

$$dZ_t^k = Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + Z_t^k(c_k - \operatorname{div} \hat{b}_k)(X_t)dt.$$

Note that both $\{V_t^k\}$ and $\{Z_t^k\}$ are semi-martingales. Applying Ito's formula, we obtain that

$$\begin{aligned} d(V_t^k Z_t^k) &= V_t^k Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + Z_t^k \nabla u_1(X_t)dM_t \\ &\quad + Z_t^k(V_t^k - u_1(X_t))(c_k - \operatorname{div} \hat{b}_k)(X_t)dt. \end{aligned}$$

Further, applying Ito's formula to Z_t^k , we get

$$\begin{aligned} &d((V_t^k + u_1(X_0))Z_t^k) \\ &= V_t^k Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + Z_t^k \nabla u_1(X_t)dM_t \\ &\quad + Z_t^k(V_t^k - u_1(X_t))(c_k - \operatorname{div} \hat{b}_k)(X_t)dt \\ &\quad + u_1(X_0)Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + u_1(X_0)Z_t^k(c_k - \operatorname{div} \hat{b}_k)(X_t)dt \\ &= (V_t^k + u_1(X_0))Z_t^k(\tilde{a}^{-1}b)^*(X_t)dM_t + Z_t^k \nabla u_1(X_t)dM_t \\ &\quad + Z_t^k(V_t^k - (u_1(X_t) - u_1(X_0)))(c_k - \operatorname{div} \hat{b}_k)(X_t)dt. \end{aligned} \tag{2.59}$$

By (2.16), (2.19), (2.56), (2.58) and [3, Theorem 2.7], there exists a subsequence $\{k_l\}$ such that $V_t^{k_l} \rightarrow u_1(X_t) - u_1(X_0)$, $t < \tau_D$, P_x -a.s. for q.e. $x \in D$ as $l \rightarrow \infty$. Therefore, (2.57) holds by (2.59).

By (2.57), we know that $\{u_1(X_{t \wedge \tau_D})Z_{t \wedge \tau_D}, t \geq 0\}$ is a P_x -local martingale for q.e. $x \in D$. We claim that $\{Z_{t \wedge \tau_D}, t \geq 0\}$ is P_x -uniformly integrable for q.e. $x \in D$. Write

$$Z_{t \wedge \tau_D} = Z_{\tau_D} 1_{\{\tau_D \leq t\}} + Z_t 1_{\{\tau_D > t\}}.$$

By (2.21), $\{Z_{\tau_D} 1_{\{\tau_D \leq t\}}, t \geq 0\}$ is P_x -uniformly integrable for q.e. $x \in D$. We now show that $\{Z_t 1_{\{\tau_D > t\}}, t \geq 0\}$ is P_x -uniformly integrable for q.e. $x \in D$. Note that for q.e. $x \in D$,

$$\begin{aligned} Z_t 1_{\{\tau_D > t\}} &\leq 1_{\{\tau_D > t\}} \exp \left(\int_0^{\tau_D} (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2} \int_0^{\tau_D} b^* \tilde{a}^{-1}b(X_s)ds \right. \\ &\quad \left. + \int_0^{\tau_D} g(X_s)ds \right) \\ &:= 1_{\{\tau_D > t\}} Z_{\tau_D}^g. \end{aligned}$$

Hence it suffices to show that $\{1_{\{\tau_D > t\}} Z_{\tau_D}^g, t \geq 0\}$ is P_x -uniformly integrable for $x \in D$.

By the strong Markov property, we get

$$\begin{aligned} 1_{\{\tau_D > t\}} E_x[Z_{\tau_D}^g | \mathcal{F}_t] &= 1_{\{\tau_D > t\}} Z_t^g E_{X_t}[Z_{\tau_D}^g] \\ &\geq 1_{\{\tau_D > t\}} Z_t^g \inf_{x \in D} E_x[Z_{\tau_D}^g] \\ &= 1_{\{\tau_D > t\}} Z_t^g \inf_{x \in D} E_x^Q \left[\exp \left(\int_0^{\tau_D} g(X_s)ds \right) \right] \\ &\geq 1_{\{\tau_D > t\}} Z_t^g. \end{aligned} \tag{2.60}$$

By (2.14) and (2.60), we obtain that $\{1_{\{\tau_D > t\}} Z_{\tau_D}^g, t \geq 0\}$ is P_x -uniformly integrable for $x \in D$. Therefore $\{Z_{t \wedge \tau_D}, t \geq 0\}$ is P_x -uniformly integrable for q.e. $x \in D$. Since u_1 is bounded continuous, we find that $\{u_1(X_{t \wedge \tau_D}) Z_{t \wedge \tau_D}, t \geq 0\}$ is a P_x -martingale for q.e. $x \in D$. Thus,

$$u_1(x) = E_x[u_1(X_{t \wedge \tau_D}) Z_{t \wedge \tau_D}], \quad \text{for q.e. } x \in D.$$

Letting $t \rightarrow \infty$, we obtain that

$$u_1(x) = E_x[f(X_{\tau_D}) Z_{\tau_D}], \quad \text{for q.e. } x \in D,$$

which proves the uniqueness. □

3 Probabilistic Representation of Non-symmetric Semigroup

In this section, we will use some techniques of Section 2 to give a probabilistic representation of the non-symmetric semigroup $\{T_t\}_{t \geq 0}$ associated with the operator L defined by (1.2). The obtained result (see Theorem 3.1 below) generalizes [15, Theorem 3.4], which is the first result on the probabilistic representation of semigroups with $\hat{b} \neq 0$, from the case of symmetric diffusion matrix A to the non-symmetric case. The methods and techniques of this paper can be applied also to some other problems such as the mixed boundary value problem, Dirichlet problem of semilinear elliptic PDEs with singular coefficients, etc. (cf. [5, 26]). We will consider them in future work.

Throughout this section, we let D be an open subset of \mathbb{R}^d , which need not be bounded. Suppose that $A(x) = (a_{ij}(x))_{i,j=1}^d$ is a Borel measurable matrix-valued function on D satisfying (1.3) and (1.4); $b = (b_1, \dots, b_d)^*$ and $\hat{b} = (\hat{b}_1, \dots, \hat{b}_d)^*$ are Borel measurable \mathbb{R}^d -valued functions on D and c is a Borel measurable function on D satisfying $|b|^2 \in L^{p \vee 1}(D; dx)$, $|\hat{b}|^2 \in L^{p \vee 1}(D; dx)$ and $c \in L^{p \vee 1}(D; dx)$ for some constant $p > d/2$. Let L and $(\mathcal{E}, D(\mathcal{E}))$ be defined as in (1.2) and (1.5), respectively. Since $|b|^2$, $|\hat{b}|^2$ and c are in the Kato class, there exists a constant $\gamma > 0$ such that $(\mathcal{E}_\gamma, D(\mathcal{E}))$ is a coercive closed form on $L^2(D; dx)$ (cf. [15, page 329]). Hence there exists a (unique) strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on $L^2(D; dx)$ which is associated with $(\mathcal{E}, D(\mathcal{E}))$. Denote by $(\mathcal{L}, D(\mathcal{L}))$ the generator of $\{T_t\}_{t \geq 0}$. Clearly \mathcal{L} is formally given by L . Denote by $\{\hat{T}_t\}_{t \geq 0}$ the dual semigroup of $\{T_t\}_{t \geq 0}$ on $L^2(D; dx)$.

We define the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ as in (1.10). Let $X = ((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$ and $\hat{X} = (X_t)_{t \geq 0}, (\hat{P}_x)_{x \in \mathbb{R}^d}$ be the Markov process and dual Markov process associated with the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ given by (1.10), respectively. Let M_t , $(\tilde{a}_{ij})_{i,j=1}^d$, v^H , etc. be defined the same as in Section 1. Denote by m the Lebesgue measure dx on \mathbb{R}^d . Now we can state the main result of this section.

Theorem 3.1. *For any $f, g \in L^2(D; dx)$, we have*

$$\begin{aligned} & \int_D f(x) T_t g(x) dx \\ = & E_m \left[f(X_0) g(X_t) \exp \left(\int_0^t (\tilde{a}^{-1} b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds \right. \right. \\ & \left. \left. + \int_0^t c(X_s) ds + N_t^{\hat{b}^H} - \int_0^t \hat{b}^H(X_s) ds \right); t < \tau_D \right]. \end{aligned} \quad (3.1)$$

Proof. By (2.1), similar to [15, Theorem 2.1], we can prove the following lemma on integrability of functionals of Dirichlet processes.

Lemma 3.2. *Suppose $f \in L^{r \vee 1}(D; dx)$ for some $r > d/2$ and $T > 0$. Then, there exists a constant $\varrho_1 > 0$ depending on f , r and T such that for any $0 \leq t \leq T$,*

$$\sup_{x \in D} E_x \left[\exp \left(\int_0^t f(X_s) ds \right); t < \tau_D \right] \leq \varrho_1 e^{\varrho_1 t},$$

and

$$\sup_{x \in D} \hat{E}_x \left[\exp \left(\int_0^t f(X_s) ds \right); t < \tau_D \right] \leq \varrho_1 e^{\varrho_1 t}.$$

We divide the proof of Theorem 3.1 into three cases.

Case 1: $\hat{b} = 0$.

For $g \in \mathcal{B}_b(D)$, we define

$$\begin{aligned} P_t g(x) := & E_x \left[\exp \left(\int_0^t (\tilde{a}^{-1} b)^*(X_s) dM_s \right. \right. \\ & \left. \left. - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds + \int_0^t c(X_s) ds \right) g(X_t); t < \tau_D \right]. \end{aligned}$$

Clearly $\{P_t\}_{t \geq 0}$ is a well-defined semigroup. We now show that $\{P_t\}_{t \geq 0}$ extends to a strongly continuous semigroup on $L^2(D; dx)$, which will be also denoted by $\{P_t\}_{t \geq 0}$.

In fact, for any $g \in L^2(D; dx)$, we obtain by Lemma 3.2 that

$$\begin{aligned} & \int_D (P_t g(x))^2 dx \\ = & \int_D \left(E_x \left[\exp \left(\int_0^t (\tilde{a}^{-1} b)^*(X_s) dM_s - \int_0^t b^* \tilde{a}^{-1} b(X_s) ds \right) \right. \right. \\ & \left. \left. \cdot \exp \left(\frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds + \int_0^t c(X_s) ds \right) g(X_t); t < \tau_D \right] \right)^2 dx \\ \leq & \int_D E_x \left[\exp \left(2 \int_0^t (\tilde{a}^{-1} b)^*(X_s) dM_s - 2 \int_0^t b^* \tilde{a}^{-1} b(X_s) ds \right) \right] \end{aligned}$$

$$\begin{aligned}
& \cdot E_x \left[\exp \left(\int_0^t b^* \tilde{a}^{-1} b(X_s) ds + 2 \int_0^t c(X_s) ds \right) g^2(X_t); t < \tau_D \right] dx \\
&= \int_D g^2(x) \hat{E}_x \left[\exp \left(\int_0^t b^* \tilde{a}^{-1} b(X_s) ds + 2 \int_0^t c(X_s) ds \right); t < \tau_D \right] dx \\
&\leq \varrho_2 e^{\varrho_2 t} \int_D g^2(x) dx
\end{aligned} \tag{3.2}$$

where $\varrho_2 > 0$ is a constant independent of g . This gives the existence of the extension of P_t to $L^2(D; dx)$. Since $C_b(D)$ is dense in $L^2(D; dx)$ and for $g \in C_b(D)$, $P_t g(x) \rightarrow g(x)$ as $t \rightarrow 0$, the continuity property of P_t follows from (3.2).

Define

$$S_t = \exp \left(\int_0^t (\tilde{a}^{-1} b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds + \int_0^t c(X_s) ds \right)$$

and

$$\bar{M}_t = \int_0^t (\tilde{a}^{-1} b)^*(X_s) dM_s.$$

Then $S_t = 1 + \int_0^t S_s d\bar{M}_s + \int_0^t S_s c(X_s) ds$. By Ito's formula, we obtain that for $u \in D(\mathcal{L})$ and $t < \tau_D$,

$$u(X_t) S_t = u(X_0) + \int_0^t S_s dM_s^u + \int_0^t u(X_s) S_s d\bar{M}_s + \int_0^t S_s \mathcal{L}u(X_s) ds.$$

Following the argument of the proof of [15, Theorem 3.2], we can show that $\{P_t\}_{t \geq 0}$ coincides with $\{T_t\}_{t \geq 0}$ for this case.

Case 2: $\hat{b} \in C_0^\infty(D)$.

Similar to the proof of [15, Theorem 3.3], we can show that for $g \in L^2(D; dx)$,

$$\begin{aligned}
& T_t g(x) \\
&= E_x \left[\exp \left(\int_0^t (\tilde{a}^{-1} b)^*(X_s) dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1} b(X_s) ds \right. \right. \\
&\quad \left. \left. + \int_0^t c(X_s) ds - \int_0^t \operatorname{div} \hat{b}(X_s) ds \right) g(X_t); t < \tau_D \right].
\end{aligned}$$

The proof of this case is complete by (2.4).

Case 3: $|\hat{b}|^2 \in L^{p \vee 1}(D; dx)$.

By Lemma 2.2(ii), we may choose a sequence $\{\hat{b}_n \in C_0^\infty(\mathbb{R}^d)\}$ such that $|\hat{b}_n - \hat{b}|^2 \rightarrow 0$ in $L^{p \vee 1}(\mathbb{R}^d; dx)$ and $\hat{b}_n^H \rightarrow \hat{b}^H$ in $H^{1,2}(\mathbb{R}^d)$ as $n \rightarrow \infty$.

Let $\{T_t^n\}_{t \geq 0}$ be the semigroup corresponding to the quadratic form \mathcal{E} with \hat{b}_n in place of \hat{b} . Then, for $f, g \in L^2(D; dx)$, we have

$$\int_D f(x) T_t^n g(x) dx$$

$$\begin{aligned}
&= E_m \left[f(X_0)g(X_t) \exp \left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s)ds \right. \right. \\
&\quad \left. \left. + \int_0^t c(X_s)ds + N_t^{\hat{b}_n^H} - \int_0^t \hat{b}_n^H(X_s)ds \right); t < \tau_D \right]. \tag{3.3}
\end{aligned}$$

By [23, Theorem 1.3], the left-hand side of (3.3) converges to $\int_D f(x)T_t g(x)dx$ as $n \rightarrow \infty$.

We will prove below that the right-hand side of (3.3) converges to the right-hand side of (3.1) as $n \rightarrow \infty$. Define for $t \geq 0$,

$$\begin{aligned}
Y_t^n &= g(X_t) \exp \left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s)ds \right. \\
&\quad \left. + \int_0^t c(X_s)ds + N_t^{\hat{b}_n^H} - \int_0^t \hat{b}_n^H(X_s)ds \right), \quad n \in \mathbb{N},
\end{aligned}$$

and

$$\begin{aligned}
Y_t &= g(X_t) \exp \left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s)ds \right. \\
&\quad \left. + \int_0^t c(X_s)ds + N_t^{\hat{b}^H} - \int_0^t \hat{b}^H(X_s)ds \right).
\end{aligned}$$

Then, the right-hand sides of (3.3) and (3.1) equal $E_{f \cdot m}[Y_t^n; t < \tau_D]$ and $E_{f \cdot m}[Y_t; t < \tau_D]$, respectively. To complete the proof, we need only show that $\{Y_t^n \mathbf{1}_{t < \tau_D}\}$ is $P_{f \cdot m}$ -uniformly integrable. We will establish this below by proving that $\sup_{n \in \mathbb{N}} E_{f \cdot m}[(Y_t^n)^2; t < \tau_D] < \infty$.

In fact, we obtain by Cauchy-Schwarz inequality that

$$\begin{aligned}
&E_{f \cdot m}[(Y_t^n)^2; t < \tau_D] \\
&= E_{f \cdot m} \left[g^2(X_t) \exp \left(2 \int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \int_0^t b^* \tilde{a}^{-1}b(X_s)ds \right. \right. \\
&\quad \left. \left. + 2 \int_0^t c(X_s)ds + 2N_t^{\hat{b}_n^H} - 2 \int_0^t \hat{b}_n^H(X_s)ds \right); t < \tau_D \right] \\
&= E_{f \cdot m} \left[g^2(X_t) \exp \left(\frac{1}{2} \int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{4} \int_0^t b^* \tilde{a}^{-1}b(X_s)ds \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \int_0^t c(X_s)ds + 2N_t^{\hat{b}_n^H} - 2 \int_0^t \hat{b}_n^H(X_s)ds \right) \right. \\
&\quad \left. \cdot \exp \left(\frac{3}{2} \int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{3}{4} \int_0^t b^* \tilde{a}^{-1}b(X_s)ds \right. \right. \\
&\quad \left. \left. + \frac{3}{2} \int_0^t c(X_s)ds \right); t < \tau_D \right] \\
&\leq E_{f \cdot m} \left[g^4(X_t) \exp \left(\int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s - \frac{1}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s)ds \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t c(X_s)ds + N_t^{4\hat{b}_n^H} - \int_0^t 4\hat{b}_n^H(X_s)ds \Big) ; t < \tau_D \Big]^{1/2} \\
& \cdot E_{f \cdot m} \left[\exp \left(3 \int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s \right. \right. \\
& \quad \left. \left. - \frac{3}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s)ds + 3 \int_0^t c(X_s)ds \right) ; t < \tau_D \right]^{1/2} \\
& = \left(\int_D f(x)T_t^{n'} g^4(x)dx \right)^{1/2} \\
& \cdot E_{f \cdot m} \left[\exp \left(3 \int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s \right. \right. \\
& \quad \left. \left. - \frac{3}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s)ds + 3 \int_0^t c(X_s)ds \right) ; t < \tau_D \right]^{1/2},
\end{aligned}$$

where $\{T_t^{n'}\}_{t \geq 0}$ is the semigroup corresponding to the quadratic form \mathcal{E} with $4\hat{b}_n$ in place of \hat{b} . Thus, we obtain by [23, Theorem 1.3] and Lemma 3.2 that

$$\begin{aligned}
& \sup_{n \in \mathbb{N}} E_{f \cdot m}[(Y_t^n)^2; t < \tau_D] \\
& \leq \sup_{n \in \mathbb{N}} \left(\int_D f(x)T_t^{n'} g^4(x)dx \right)^{1/2} \cdot E_{f \cdot m} \left[\exp \left(3 \int_0^t (\tilde{a}^{-1}b)^*(X_s)dM_s \right. \right. \\
& \quad \left. \left. - \frac{3}{2} \int_0^t b^* \tilde{a}^{-1}b(X_s)ds + 3 \int_0^t c(X_s)ds \right) ; t < \tau_D \right]^{1/2} \\
& < \infty.
\end{aligned}$$

□

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